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Dark solitary waves in the nonlinear Schrödinger equation with third order dispersion, self-steepening, and self-frequency shift

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We solve the higher order nonlinear Schrödinger equation describing the propagation of ultrashort pulses in optical fibers. By means of the coupled amplitude-phase formulation fundamental (solitary wave) dark soliton solutions are found. [S1063-651X(99)51407-4]

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Soliton phenomena can be used to construct stable short (subpicosecond) pulses if the peak power and minimum dispersion are properly chosen. Since solitons are formed by the balance of dispersion and nonlinearity, to construct a soliton with the least width for a given peak power, it is desirable to launch a pulse at a wavelength near the zero group dispersion. It is under these conditions that the classical nonlinear Schrödinger (NLS) equation fails in the physical description of the propagation of optical pulses in fibers. Therefore, higher order terms, such as third order dispersion and Raman effects, become important and must be included. The nonlinear propagation equation for these pulses (femtosecond pulses) was derived by Kodama and Hasegawa [1] and is quite different from the well known NLS equation. Unfortunately, in general, it is not completely integrable. However, for some given specific conditions, this equation provides analytic solitary wave solutions [2].

In this Rapid Communication we are concerned with the existence of dark solitary waves [3]. Dark solitons are generally considered to be less desirable for applications in high speed communication systems because of their higher average power and resulting undesirable effects, such as excitation of the stimulated Brillouin backscattering. On the other hand, bright solitons have a drawback in that they have difficulty in fully utilizing the line capacity because of the necessity of keeping relatively large separations between solitons to avoid accumulation of bit rate error. Fiber loss causes a decrease in the amplitude of a bright soliton, along with a corresponding increase in the width. This effect is smaller in the dark soliton case. It was shown both numerically and analytically that the time jitter in a dark soliton is lower than the corresponding one of a bright soliton [4,5]. The interaction force between two dark solitons is always repulsive, unlike the bright soliton case, and decreases twice as fast as a function of the distance between the solitons. The separation increases monotonically rather than periodically as in the case of bright solitons. To date, no analytical solution for dark solitary waves has been provided, and all of these features cited above make these kinds of solitons very physically interesting.

The higher order nonlinear Schrödinger (HONLS) equation describing the propagation of femtosecond pulses in optical fibers can be written in the form

$$q_z = ia_1 q_{tt} + ia_2|q|^2 q + a_3 q_{ttt} + a_4(|q|^2 q)_t + a_5 q(|q|^2)_t, \tag{1}$$

where the terms on the right-hand side are the group velocity dispersion (GVD), self-phase modulation (SPM), the term proportional to a_3 results from including the cubic term in the expansion of the propagation constant. This term includes the effects of third order dispersion (TOD) that become important for ultrashort pulses because of their wide bandwith even when the wavelength is relatively far away from the zero-dispersion point [6]. The term proportional to a_4 results from including the first derivative of the slowly varying part of the nonlinear polarization. It is responsible for self-steepening and shock formation at a pulse edge. The last term proportional to a_5 has its origin in the delayed Raman response and is responsible for the self-frequency shift, a phenomenon first discovered by Mitschke and Mollenauer [7]. When the last three are negligible, the equation becomes the NLS equation, which is completely integrable by the inverse scattering transform [8]. When the constant $a_5=0$, the resulting equation, is called the modified nonlinear Schrödinger (MNLS) equation. In general, the MNLS equation, including the self-steepening term, has been analytically solved [9]. Many works have dealt with the Painlevé analysis and the conditions for integrability [10,11], even reporting on the Lax pair and *N*-soliton solutions [12]. Other analyses have provided solitary wave solutions by means of traveling wave methods [13]. These solutions are all symmetric and the natural asymmetry due to self-steepening that leads to shock formation is considered in [14]. However, when the last term proportional to a_5 is not ignored, the propagation equation becomes more complex and the question of the existence of analytic solutions arises one more time.

To provide an answer to this, let us scale Eq. (1) as in Ref. [2] in the forms

$$q = b_1\psi, \quad z = b_2\xi, \quad t = b_3\tau,$$

and choose b_1, b_2, b_3 so the coefficients corresponding to GVD, SPM, and TOD become unity. Thus, Eq. (1) may be written as

$$\psi_\xi = i\psi_{\tau\tau} + i|\psi|^2\psi + \psi_{\tau\tau\tau} + c_1(|\psi|^2\psi)_\tau + c_2\psi(|\psi|^2)_\tau, \tag{2}$$

with $c_1 = a_1a_4/a_2a_3$ and $c_2 = a_1a_5/a_2a_3$. We begin our analysis assuming a solution given by the expression

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$$\psi(\xi,\tau)=P(\tau+\beta\xi)\exp[i(\kappa\xi-\omega\tau)]$$
$$=P(\chi)\exp[i(\kappa\xi-\omega\tau)], \quad (3)$$

where the function P must be real. Substituting Eq. (3) into Eq. (2) and removing the exponential term, we obtain

$$\beta P_\chi+i\kappa P=P_{\chi\chi\chi}+i(1-3\omega)P_{\chi\chi}+(2\omega-3\omega^2+3c_1P^2+2c_2P^2)P_\chi+i(1-c_1\omega)P^3+i(\omega^3-\omega^2)P.$$

Separating the real and imaginary parts, we have

$$\beta P_\chi=P_{\chi\chi\chi}+(2\omega-3\omega^2+3c_1P^2+2c_2P^2)P_\chi, \quad (4)$$

$$\kappa P=(1-3\omega)P_{\chi\chi}+(1-c_1\omega)P^3+(\omega^3-\omega^2)P. \quad (5)$$

Equation (4) only contains third order and first order derivatives. We can, therefore, write it in the form

$$P_{\chi\chi\chi}=(\beta-2\omega+3\omega^2-3c_1P^2-2c_2P^2)P_\chi,$$

and it is possible to integrate it to give

$$P_{\chi\chi}=(\beta-2\omega+3\omega^2)P-c_1P^3-\frac{2}{3}c_2P^3. \quad (6)$$

If we express Eq. (5) in the form

$$P_{\chi\chi}=\frac{\kappa+\omega^2-\omega^3}{1-3\omega}P+\frac{c_1\omega-1}{1-3\omega}P^3, \quad (7)$$

Eqs. (6) and (7) will be equivalent, provided that

$$\beta-2\omega+3\omega^2=\frac{\kappa+\omega^2-\omega^3}{1-3\omega},$$

$$\frac{3c_1+2c_2}{3}=\frac{1-c_1\omega}{1-3\omega}.$$

The latter can be written as

$$\omega=\frac{3c_1+2c_2-3}{6(c_1+c_2)}, \quad (8)$$

and the former gives

$$\kappa=(\beta-2\omega+3\omega^2)(1-3\omega)-\omega^2+\omega^3. \quad (9)$$

Thus, we get the ordinary differential equation

$$P_{\chi\chi}=(\beta+3\omega^2-2\omega)P-\frac{3c_1+2c_2}{3}P^3, \quad (10)$$

which coincides with the evolution of an anharmonic oscillator with potential:

$$U(P)=-\frac{1}{2}(\beta+3\omega^2-2\omega)P^2+\frac{1}{12}(3c_1+2c_2)P^4.$$

Now we proceed with the coupled amplitude-phase formulation [15]. Equation (10) thus becomes

$$P_{\chi\chi}=\frac{d}{dP}\left[\frac{\beta+3\omega^2-2\omega}{2}P^2-\frac{3c_1+2c_2}{12}P^4\right].$$

Since

$$P_{\chi\chi}=\frac{d}{dP}\left[\frac{1}{2}(P_\chi)^2\right]$$

we can then write

$$d\chi=\left[(\beta+3\omega^2-2\omega)P^2-\frac{3c_1+2c_2}{6}P^4+2E\right]^{-1/2}dP, \quad (11)$$

where E is an arbitrary constant of integration, which coincides with the energy of the anharmonic oscillator.

Integrating Eq. (11) for different values of E, we get the amplitude function $P(\chi)$. It is very interesting to look carefully at the above equation. Formally, it is identical to Eq. (17) in [15]. Thus it is not inconceivable that both present the same solutions. If we set $E=0$ in Eq. (11) we obtain the bright optical solitary wave solution provided that $\beta+3\omega^2-2\omega$ and $3c_1+2c_2$ be positive quantities, just as in [2]. However, if we look for a value of E, such as the expression inside the square root be a perfect square, we can also obtain the dark optical solitary wave solution provided that $\beta+3\omega^2-2\omega$ and $3c_1+2c_2$ be negative quantities in contrast with [2]. This energy value is

$$E=-\frac{3}{4}\frac{(\beta+3\omega^2-2\omega)^2}{3c_1+2c_2}.$$

In this case the solution for the amplitude function is

$$P(\chi)=\left[\frac{3(\beta+3\omega^2-2\omega)}{3c_1+2c_2}\right]^{1/2}\tanh\left[-\left(\omega-\frac{3}{2}\omega^2-\frac{1}{2}\beta\right)^{1/2}\chi\right],$$

which is the soliton power and width, respectively:

$$P_0=\frac{3(\beta+3\omega^2-2\omega)}{3c_1+2c_2}, \quad (12)$$

$$T_0=\sqrt{\frac{2}{2\omega-3\omega^2-\beta}}. \quad (13)$$

So, for typical experimental values in ''standard'' optical fibers (see [16] and the references therein), we obtain for pulses with $T_0\approx 17$ fs, propagating at a wavelength of 1 μm with $\beta_2\approx 20$ ps^2 km^{-1} and $\beta_3\approx 0.08$ ps^3 km^{-1}, the values for the dimensionless constants $c_1\approx -1.58$ or $c_1\approx -0.79$ (depending on the index of refraction in the fiber, see [16]) and $c_2\approx -0.5$. By substituting this data into Eq. (8) we have $\omega\approx 0.7$ or $\omega\approx 0.8$. Then, given T_0 and ω, it is straightforward to calculate β with the help of Eq. (13), resulting in $\beta\approx -7000$. Finally, taking these values for β, ω, c_1, and c_2, and evaluating Eq. (12), the power required for generating the dark pulses arises, as $P_0\approx 3.6$ kW or $P_0\approx 6$ kW, for the two cases considered above, respectively.

In conclusion, we have reported the fundamental dark soliton solution for the HONLS equation, which always exists provided that certain relations between the parameters are fulfilled. These relations are in contrast with those found in [2], which only provide for bright solitary waves.

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